PMM U.S.S.R., Vol.53, No.5, pp.680-683, 1989
Printed in Great Britain
0021-8928/89 $10.00+0.00

ON THE SHAPE OF A MINIMUM RESISTANCE SOLID OF ROTATION PENETRATING INTO PLASTICALLY COMPRESSIBLE MEDIA WITHOUT DETACHMENT*

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In /1/, a variational problem was formulated concerning the shape of the minimum resistance of thin bodies penetrating into compressible and viscoplastic media which simulated soils and metals, respectively. This problem was formulated under the assumption that the hypothesis of planar cross-sections holds. The solution was analysed and the optimal solids of rotation penetrating into plastically compressible media were found when there is friction and when one of the geometrical parameters defining the shape of the body is specified.

The more-complex problem of the optimal shape of a solid of rotation when two of the geometrical parameters defining its shape are simultaneously specified is solved below subject to the same assumptions.

1. Formulation of the problem. The resistance D acting on a thin fine-pointed solid of rotation of length $x_k = L$ along the x-axis and with a generatrix

$$y = f(x), \quad 0 \leqslant f'(x) \leqslant k_0 = \text{const}, \quad f''(x) < 0, \quad f(0) = 0 \tag{1.1}$$

which penetrates at a constant velocity u into a plastically compressible medium is given by the functional

$$D = 2\pi \int_0^{x_k} p_0 (f' + \mu_0) f\, dx, \quad p_0 = Af'^2 + Bff'' + G = \beta^2 \tag{1.2}$$

where p_0 is the pressure of the medium on the surface of the penetrating body, μ_0 is the coefficient of dry friction and $A \geqslant 0$, $B \geqslant 0$ and $G \geqslant 0$ depend on the properties of the medium /1, 2/.

In a number of problems, where both the linear dimensions as well as the internal profiles of the penetrating body are important, it is advisable to specify its shape by giving two of the geometrical parameters, the choice of which depends on the constructional requirements. It follows /1/ that, in doing this, it is best to select the maximum diameter of the body, $d = 2R_0$, as one of these parameters while the length of the body, L, its volume V, or the lateral surface area S, can be taken as the other.

2. Shape of the minimum resistance body when its length L and maximum diameter d are specified. The solution of the problem is sought in the class of functions $f(x)$ with clamped ends:

$$f(0) = 0, \; f(L) = d/2 = R_0.$$

Since the volume of the body V and the surface area S are arbitrary, then, as was shown in /1/ by a detailed analysis of the conditions for a minimum, the equation

$$\Phi(f') f = -C_1 + \lambda_4' f' - \lambda_5 k_0 + \lambda_6 G + \lambda_6' Bff' - \lambda_6 f'^2 (A - B), \quad k_0 = f' + \alpha^2 \tag{2.1}$$
$$\Phi(f') = 2(A - B) f'^3 + \mu_0 (A - 2B) f'^2 - \mu_0 G$$

must be satisfied along each of the arcs from which the extremal may be constructed, where $\lambda_4(x), \lambda_5(x)$ and $\lambda_6(x)$ are Lagrange multipliers. The necessary conditions for a minimum have the form /1/

$$\lambda_4(x) \geqslant 0, \quad \lambda_5(x) \geqslant 0, \quad \lambda_6(x) \leqslant 0 \tag{2.2}$$

If, along the arcs of the extremum, $\lambda_4 \neq 0$ and $\lambda_5 \neq 0$ then $\lambda_6 = 0$ and, vice versa, if $\lambda_6 \neq 0$ then $\lambda_4 = \lambda_5 = 0$ It has been shown that the functions $\lambda_4(x)$ and $\lambda_6(x)$ are continuous along the optimal contour and

*Prikl. Matem. Mekhan., 53, 5, 860-863, 1989

$$\lambda_4(x_c) = \lambda_6(x_c) = 0, \ \lambda_4(0) = \lambda_6(0) = 0 \tag{2 3}$$

where x is a point where the arcs of the extremal join. The resistance on the optimal body in the case under consideration is determined by the formula /1/

$$C_p = -{}^1/_2 C_1 (R_0/f_k' - L) \tag{2.4}$$

Let us investigate from which arcs an extremal can be constructed and what is their sequence.

It has been shown in /1/ that, regardless of the conditons which are specified, the extremal always terminates with an arc of zero pressure, $p_0 = 0$. It cannot begin with this arc. The extremal also cannot begin with an arc $f'' \neq 0, \beta \neq 0$ since, in this case, it would be expected that $\lambda_4 = \lambda_5 = \lambda_6 \equiv 0$ while it follows that $f'(0) = \infty$ when $C_1 \neq 0$ and $f(0) = 0$. This contradicts (1.1) and, consequently, the extremal must begin from the arc $f'' = 0$.

Since the body is convex, $(R_0/f_k') > L$ everywhere and, since $C_p > 0$, it follows from (2.4) that $C_1 \leqslant 0$. However, $C_1 = 0$ only if $f_k' = 0$ and, when this is so, the extremal consists of two arcs $f = kx$ and $p_0 = 0$. If $C_1 \neq 0$, then the extremal can be constructed from the three arcs

$$1)\ f'' = 0, \quad 2)\ f'' \neq 0,\ \beta = p_0^{1/2} = (Af'^2 + Bff'' + G)^{1/2} \neq 0, \quad 3)\ \beta^2 = p_0 = 0$$

Theorem 1. Under the conditions of the problem being considered, there is a uniquely possible sequence of the extremal arcs, *1*, *2* and *3*.

Initially, we will show that arc *3* can only be located at the end of the extremal. The condition

$$f' = [(C_0/f)^\gamma - G/A]^{1/2} = z^{1/2}, \quad z > 0, \quad \gamma = 2A/B \tag{2.5}$$

must be satisfied along arc *3*, where f' is the solution of the equation and C_0 is an integration constant.

Let us consider the function

$$F_0(f) = \Phi(f') + C_1/f = F_1(z) = 2(A - B) z^{3/2} + \mu_0 (A - 2B) z + (C_1/C_0)(z + G/A)^{1/\gamma} - \mu_0 G \tag{2.6}$$

Allowing for the fact that $C_1 \leqslant 0$, $z > 0$ and $A - B > 0$ and $A - 2B < 0$ always in the case of the media being considered, we obtain that $d^2F_1/dz^2 > 0$. Since $F_1(0) < 0$ and $F_1(z) \to +\infty$ when $z \to \infty$, $F_1(z)$ has a unique minimum and, consequently, there exists a unique value $f = f_1$ for which $F_0(f_1) = 0$.

Lemma 1. Arc *2* cannot be located after arc *3*. Let us assume that this is not so. then, according to (2.2) and (2.6), the condition

$$\Phi(f_{c2}') f_{c2} = -C_1, \quad F_0(f_{c2}) = 0 \tag{2.7}$$

must be satisfied at the point where the arcs join, where f_{c2} is the ordinate of the point where arc *3* joins arc *2*.

The extremal cannot begin with arc *3* since it is mandatory that the arc $\beta \neq 0$ should come before arc *3* and since, when it joins with it at the point (x_{c1}, f_{c1}) we have $\lambda_6(x_{c1}) = 0$, $\lambda_6'(x_{c1}) \leqslant 0$, then

$$[\Phi(f_{c1}'^+) f_{c1}^+ + C_1] \leqslant 0, \quad F_0(f_{c1}) \leqslant 0 \tag{2.8}$$

Since $f_{c1} < f_{c2}$, then $z_{c1} > z_{c2}$, it follows from (2.6) that $F_0(f_{c1}) = F_1(z_{c1}) \leqslant 0$ and $F_1(z) \leqslant 0$ for all z in the interval $z < z_{c1} \leqslant z_1$ and this means that $F_0(f_{c2}) = F_1(z_{c2}) < 0$ which contradicts (2.7).

Lemma 2. Arc *1* cannot be located after arc *3*.

Let us suppose that this is not so. Then, as follows from (2.2), the relationship

$$\lambda_4'(x) = \Phi(k)\, x + \Phi(k)\, C_2/k + C_1/k \tag{2.9}$$

must be satsfied the arc $f = kx + C_2$, where $C_2 > 0$

In order that conditions (1.4) and (1.5) should be satisfied while taking account of the fact that $\lambda_4(x)$ and $\lambda_6(x)$ are continuous functions along the whole of the extremal, it is necessary that $\Phi(k) < 0$. However, then, since $C_1 \leqslant 0$, $\lambda_4'(x) < 0$ always and this cannot be satisfied.

By generalizing the results of Lemmas 1 and 2, we get that arc *3* can only be located on the last part of the extremal. As has already been mentioned, the extremal begins with arc *1*. Arc *2* can then follow. In an analogous manner to the proof of Lemma 2 which has just been carried out, it can be shown that arc *1* cannot be located after arc *2*.

The theorem is thereby proved.

As will be shown below, cases are possible when arc *2* is missing from the extremal but the order of arcs *1* and *3* is preserved when this occurs.

The extremal always begins with arc *1*. In its turn, two versions are possible for arc *1*:

a) $f = k_0 x$, b) $f = kx$, $(k_0 > k)$ If the extremal begins with arc b, then $\lambda_5 \equiv 0$, and, from (2.2) when $f = 0$, we get

$$\lambda_4'(0)k = C_1 \leqslant 0 \tag{2.10}$$

Subject to the conditions $\lambda_4(0) = 0$, the equality (2.4) is only satisfied when $C_1 = 0$ Then, $\lambda_4(x) \equiv 0$ and k is determined from the equation $\Phi(k) = 0$ When this is so, $f_k' = 0$ and there can be no arc *2* in the extremal. Now let the extremal begin with arc a. Arc *2* can follow after arc *1* only in the case when $\Phi(k_0) > 0$. However, values of μ_0 exist for which

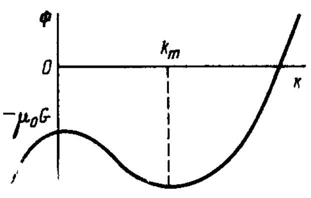

Fig.1

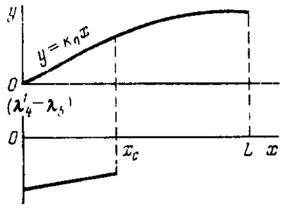

Fig.2

this condition will be violated, that is, at fairly high coeffcients of friction, μ_0, and, regardless of the specification of the values of L and d, the extremal will consists of just two arcs: *1* and *3*.

Let us show that, while such a value μ_0^* will exist, the extremal must contain arc *2* when $\mu_0 < \mu_0^*$ regardless of the specification of L and d. We will use the notation $k_1 = R_0/L$ and consider μ_0 such that $\Phi(k_1) > 0$ In this case the extremal cannot begin with arc b since $k > k_1$ Meanwhile, since $k_0 > k_1$, it follows from the general form of the function $\Phi(k)$ (Fig.1) that $k_m = {}^1/_3\mu_0(2B - A)/(A - B)$ and $\Phi(k_0) > 0$, which assumes the possibility of the existence of arc 2. Let this not be so, that is, let arc 3 follow after arc a for all μ_0 when $\Phi(k_1) > 0$. On joining the arcs at the point (x_c, f_c), we obtain the condition

$$(\lambda_4' - \lambda_5)(x_c) \leqslant 0 \tag{2.11}$$

from (2.1)-(2.3).

The function $\lambda_4' - \lambda_5$ is linear on arc a (Fig.2)

$$(\lambda_4' - \lambda_5)(x) = \Phi(k_0)x + C_1/k_0 \tag{2.12}$$

Since it has been assumed that there is no arc *2*, the shape of the extremal for specified L and d does not change as μ_0 is varied. However, then, since $(dC_p/d\mu_0) > 0$, we get that $(d(-C_1)/d\mu_0) > 0$ and

$$d[(\lambda_4' - \lambda_5)(x_c)]/d\mu_0 < 0$$

From (2.11) and (2.12), since $\Phi(k_0) > 0$, we find μ_0^*:

$$(\lambda_4' - \lambda_5)(x_c) = 0$$

When $\mu_0 < \mu_0^*$, condition (2.11) will be violated, that is, in this case we get a contradiction with the assumption that arc *2* does not exist. It can be seen that the value of μ_0^* is determined from the system of equations

$$k_0((R_0/f_k') - L)\Phi(k_0) = (Ak_0^2 + G)(\mu_0 + k_0)f_c$$

$$L = \int_{f_c}^{R_0} \frac{df}{[(C_0/f)^\gamma - G/A]^{1/2}} + \frac{f_c}{k_0}$$

$$C_0 = f_c(k_0^2 + G/A)^{1/\gamma}, \quad f_k = [(C_0/R_0)^\gamma - G/A]$$

If the value which has been found $\mu_0^* > 0$ then, when $0 \leqslant \mu_0 \leqslant \mu_0^*$, the extremal consists of three arcs while, when $\mu_0 > \mu_0^*$, it consists of two arcs. If, for the specified values of L and d, the value $\mu_0^* < 0$ then, in this case, the extremal consists of two arcs and is independent of any change in the coefficient of friction μ_0

Using the results of the analysis of the necessary conditions which has been carried out above, let us write out the system of equations for finding the parameters which define the arcs of the extremal when $\mu_0 < \mu_0^*$ in explicit form.

Let f_{c1} and f_{c2} be the ordinates of the points of the transitions from arc *1* onto arc *2* and from arc *2* onto arc *3* respectively and let f_{c2}' be the tangent of the angle between the tangent to the contour with the ordinate f_{c2} and the x-axis. In order to determine these parameters and the values of C_1, C_0 and f_k' which make it possible to obtain the shape of the extremal in an explicit form when L and $d = 2R_0$ are specified, we have the system of equations

$$\Phi(k_0) f_{c1} = -C_1, \quad \Phi(f_{c2}') f_{c2} = -C_1$$
$$L = f_{c1}/k_0 + I_1(f_{c1}, f_{c2}) + I_2(f_{c2}, R_0) \tag{2.13}$$
$$I_n(\alpha_1, \alpha_2) = \int_{\alpha_1}^{\alpha_2} x_n'(f)\, df \quad (n = 1, 2)$$

$$R_0 = C_0/[f_k'^2 + G/A]^{1/\gamma}, \quad f_{c2} = C_0/[f_{c2}'^2 + G/A]^{1/\gamma}$$
$$(-C_1)[(R_0 - Lf_k')/f_k'] = \int_0^L (Af'^2 + Bff'' + G)(f' + \mu_0) f\, dx$$

Here,

$$x_1'(f) = \chi_+^{1/3} + \chi_-^{1/3}, \quad \chi_\pm = -q/2 \pm \sqrt{q^2/4 + p^3/27}$$
$$q = 2(A - B) fN, \quad p = (2B - A)\mu_0 fN, \quad N = 1/(\mu_0 Gf - C_1), \quad x_2'(f) = \tag{2.14}$$
$$[(C_0/f)^\gamma - G/A]^{1/2}$$

When the coefficients of the medium A, B, G, μ_0 are known and the magnitudes of R_0 and L are specified, the system of six Eqs.(2.13) uniquely defines the values of $f_{c1}, f_{c2}, f_{c2}', C_1, f_k'$ and C_0 which, in their turn from (2.14), uniquely define the segments of the arcs of the extremal:

1) $f = k_0 x$ when $f \leqslant f_{c1}$,

2) $x = f_{c1}/k_0 + I_1(f_{c1}, f)$ when $f_{c1} \leqslant f \leqslant f_{c2}$,

3) $x = f_{c1}/k_0 + I_1(f_{c1}, f_{c2}) + I_2(f_{c2}, f)$ when $f_{c2} \leqslant f \leqslant L$

When $\mu_0 > \mu_0^*$, the extremal consists of two arcs for which the joining coordinates f_c and C_0 are found from the system of equations

$$L = f_c/k + I_2(f_c, R_0), \quad f_c = C_0/[k^2 + G/A]^{1/\gamma}$$

where, in its turn, and regardless of the actual values of L and d, the coefficient k is either equal to k_0 or is determined from the equation $\Phi(k) = 0$.

Hence, we have carried out a detailed investigation of the shape of the minimum resistance body penetrating into plastically compressible media as a function of the coefficient of friction of the medium around the surface of the body.

REFERENCES

1. BUNIMOVICH A.I. and YAKUNINA G.E., On the shape of minimum resistance solids of rotation moving in plastically compressible and viscoplastic media, Prikl. Matem. i Mekh., 51, 3, 1987.
2. RAKHMATULIN KH.A., SAGOMONYAN A.YA. and ALEKSEYEV N.A., Problems of Soil Dynamics, Izd. Moskovsk. Gos. Univ., Moscow, 1974.

Translated by E.L.S.